Approximating Curves by Polynomials

Why do we care about polynomial approximations?

Polynomials are easy to work with - we can differentiate them, integrate them, and evaluate them using only basic arithmetic. But many important functions like e^x, $\sin x$, and $\ln(1+x)$ are not polynomials. If we could approximate these functions with polynomials, calculations would become much simpler.

Example

Find an approximation to $f(x) = \sqrt{1+x}$ near $x = 0$.

Let's build increasingly accurate polynomial approximations.
Linear approximation: *We want a line through $(0,1)$ with the same gradient as $f(x)$ at $x = 0$.*
Since $f(0) = 1$ and $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, so $f'(0) = \frac{1}{2}$:

$$p_1(x) = 1 + \frac{1}{2}x$$

Quadratic approximation: *Now let's match the value, first derivative, AND second derivative at $x = 0$.*
Since $f''(x) = -\frac{1}{4}(1+x)^{-3/2}$, so $f''(0) = -\frac{1}{4}$.
If $p_2(x) = a + bx + cx^2$, then:

- $p_2(0) = a = 1$
- $p_2'(0) = b = \frac{1}{2}$
- $p_2''(0) = 2c = -\frac{1}{4}$, *so* $c = -\frac{1}{8}$

Therefore: $p_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$

Notice that $p_2(x)$ fits $f(x)$ better than $p_1(x)$ near $x = 0$. The quadratic term allows the polynomial to curve, matching not just the slope but also the rate of change of the slope.

Definition (Agreement to the nth degree)**.** Two functions f and g **agree to the nth degree** at $x = 0$ if:

$$f(0) = g(0), \quad f'(0) = g'(0), \quad f''(0) = g''(0), \quad \ldots, \quad f^{(n)}(0) = g^{(n)}(0)$$

The Maclaurin Polynomial Formula

Theorem (Maclaurin Polynomial)

The polynomial of degree n which agrees with $f(x)$ to the nth degree at $x = 0$ is:

$$p_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n = \sum_{r=0}^{n} \frac{f^{(r)}(0)}{r!}x^r$$

Why the factorials? *When we differentiate x^r exactly r times, we get $r!$. So to make the rth derivative of our polynomial equal to $f^{(r)}(0)$, we need the coefficient of x^r to be $\frac{f^{(r)}(0)}{r!}$.*

Suppose we have $p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, then we must have

$$\begin{aligned}
p_n(0) &= a_0 &&= f(0) \\
p_n'(0) &= a_1 &&= f'(0) \\
p_n''(0) &= 2a_2 &&= f''(0) \\
p_n^{(3)}(0) &= 3 \cdot 2a_3 &&= f^{(3)}(0) \\
&\vdots \\
p_n^{(k)}(0) &= k! && a_k = f^{(k)}(0)
\end{aligned}$$

Example

Find the Maclaurin polynomial of degree 4 for $f(x) = (1+x)^{-3}$.

The derivatives are:

$$\begin{aligned} f(x) &= (1+x)^{-3} & f(0) &= 1 \\ f'(x) &= -3(1+x)^{-4} & f'(0) &= -3 \\ f''(x) &= 12(1+x)^{-5} & f''(0) &= 12 \\ f'''(x) &= -60(1+x)^{-6} & f'''(0) &= -60 \\ f^{(4)}(x) &= 360(1+x)^{-7} & f^{(4)}(0) &= 360 \end{aligned}$$

Therefore:

$$p_4(x) = 1 + \frac{(-3)}{1!}x + \frac{12}{2!}x^2 + \frac{(-60)}{3!}x^3 + \frac{360}{4!}x^4 = 1 - 3x + 6x^2 - 10x^3 + 15x^4$$

Example

Find the Maclaurin polynomial of degree 4 for $f(x) = \sin(2x)$.

From Polynomials to Series

Definition (Maclaurin Series)**.** The **Maclaurin series** of $f(x)$ is the infinite series:

$$f(x) = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!}x^r = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

[The set of x values for which this series converges to $f(x)$ is called the **interval of validity**.]

Standard Maclaurin Series

The Exponential Function

Example

Find the Maclaurin series for $f(x) = e^x$.

Since $\frac{d}{dx}e^x = e^x$, all derivatives of e^x equal e^x.
Therefore: $f^{(r)}(0) = e^0 = 1$ for all $r \geq 0$.
The Maclaurin series is:

$$e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Valid for all $x \in \mathbb{R}$.

Trigonometric Functions

Example

Find the Maclaurin series for $\sin x$

For $f(x) = \sin x$:

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

The pattern repeats with period 4. Only odd derivatives are non-zero:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+1}}{(2r+1)!}$$

Both valid for all $x \in \mathbb{R}$.

The Natural Logarithm

Example

Find the Maclaurin series for $f(x) = \ln(1+x)$.

The derivatives are:

$$
\begin{aligned}
f(x) &= \ln(1+x) & f(0) &= 0 \\
f'(x) &= (1+x)^{-1} & f'(0) &= 1 \\
f''(x) &= -(1+x)^{-2} & f''(0) &= -1 \\
f'''(x) &= 2(1+x)^{-3} & f'''(0) &= 2 \\
f^{(4)}(x) &= -6(1+x)^{-4} & f^{(4)}(0) &= -6
\end{aligned}
$$

The general pattern: $f^{(r)}(0) = (-1)^{r-1}(r-1)!$
Therefore: $\frac{f^{(r)}(0)}{r!} = \frac{(-1)^{r-1}(r-1)!}{r!} = \frac{(-1)^{r-1}}{r}$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} x^r$$

Valid for $-1 < x \leq 1$.

Fact (Standard Maclaurin Series) —

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots \qquad \text{for } x \in \mathbb{R}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \cdots \qquad \text{for } x \in \mathbb{R}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^r \frac{x^{2r}}{(2r)!} + \cdots \qquad \text{for } x \in \mathbb{R}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{r-1} \frac{x^r}{r} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \qquad \text{see below}$$

For $(1+x)^n$: valid for all x if $n \in \mathbb{N}$, otherwise for $|x| < 1$.

Applications of Maclaurin Series

Numerical Calculations

Example

Find e to 4 decimal places.

Substitute $x = 1$ into the expansion for e^x:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

Calculating term by term:

$$\begin{aligned} &= 1 + 1 + 0.5 + 0.166\,667 + 0.041\,667 + 0.008\,333 + 0.001\,389 + \cdots \\ &\approx 2.7183 \end{aligned}$$

Example (WJEC) (a) Show that the first two non-zero terms in the Maclaurin expansion of $\sin^{-1} x$ are given by

$$\sin^{-1} x = x + \frac{x^3}{6} + \cdots$$

(b) By writing $x = \frac{1}{2}$, deduce an approximation to π as a rational fraction in its lowest terms.

(c) The equation $\sin^{-1} x = 1.002x$ is satisfied by a small positive value of x. Find an approximation to this value, giving your answer correct to three decimal places.

Composite Functions

Example

Find the Maclaurin expansion of $e^{2x}\cos 3x$ up to the term in x^4.

Using the standard expansions with x replaced by 2x and 3x respectively:

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \cdots$$
$$\cos 3x = 1 - \frac{9x^2}{2} + \frac{27x^4}{8} + \cdots$$

Multiplying:

$$\begin{aligned} e^{2x}\cos 3x &= \left(1 + 2x + 2x^2 + \frac{4x^3}{3} + \cdots\right)\left(1 - \frac{9x^2}{2} + \cdots\right) \\ &= 1 + 2x + \left(2 - \frac{9}{2}\right)x^2 + \left(\frac{4}{3} - 9\right)x^3 + \left(\frac{2}{3} - 9 + \frac{27}{8}\right)x^4 + \cdots \\ &= 1 + 2x - \frac{5x^2}{2} - \frac{23x^3}{3} - \frac{119x^4}{24} + \cdots \end{aligned}$$

Tip

When multiplying series, be systematic! Write out both series, then collect terms of the same degree. Remember that you only need terms up to the required power.

Intervals of Validity

Fact — The interval of validity is the set of x values for which the Maclaurin series converges to the function value. Key points:

- For e^x, $\sin x$, $\cos x$: valid for all real x
- For $\ln(1+x)$: valid for $-1 < x \leq 1$
- For $(1+x)^n$ with $n \notin \mathbb{N}$: valid for $|x| < 1$
- When composing functions, check that the inner function stays within the validity range

Example

For what values of x is the Maclaurin series for $\ln(1+3x^2)$ valid?

We substitute $u = 3x^2$ into $\ln(1+u)$, which requires $-1 < u \leq 1$. Since $u = 3x^2 \geq 0$, we need $3x^2 \leq 1$, ie $x^2 \leq \frac{1}{3}$. Therefore the series is valid for $-\frac{1}{\sqrt{3}} \leq x \leq \frac{1}{\sqrt{3}}$.

Example (OCR June 2010 Q3)

Given that the first three terms of the Maclaurin series for $(1 + \sin x)e^{2x}$ are identical to the first three terms of the binomial series for $(1 + ax)^n$, find the values of the constants a and n